Isotropic Turbulence

Strictly speaking, isotropy, *i.e.* independence of orientation (*riktning*), implies homogeneity, *i.e.* independence of position in space. In most situations, all the averaged properties of isotropic turbulence can also be assumed to be invariant under reflection in space.

Notation

A number of different notations are used for the mean velocity and the fluctuating part of the velocity in turbulent flows. In this chapter, a fluctuation, *i.e.* the difference between an instantaneous physical quantity and its mean, is denoted by a lower-case letter, *e.g.* u. Some alternative notations are given in the following table.

	inst.	mean	fluct.
previous chapters	u	U	u'
this chapter	—	U	u
Pope (2000)	U	$\langle U\rangle$	u
Tennekes & Lumley	$\widetilde{u}$	U	u

The operation of forming an average will be denoted by angular brackets, $\langle u\rangle = 0$.

A. Singlepoint and Two-Point Velocity Correlations

Singlepoint correlations

In isotropic turbulence, the singlepoint correlations satisfy,

$$\left\langle u^2\right\rangle \;=\; \left\langle v^2\right\rangle \;=\; \left\langle w^2\right\rangle \qquad \text{while} \qquad \langle uv\rangle \;=\; \langle vw\rangle \;=\; \langle wu\rangle \;=\; 0\,,$$

so that we can define $u_0^2 = \left\langle u^2\right\rangle = \frac{2}{3}K$. ($u_{\mathrm{T}} \simeq u_0$.) In Cartesian coordinates, we can write,

$$\left\langle u_1^2\right\rangle \;=\; \left\langle u_2^2\right\rangle \;=\; \left\langle u_3^2\right\rangle \;=\; u_0^2 \qquad \text{with} \qquad \langle u_1u_2\rangle \;=\; \langle u_2u_3\rangle \;=\; \langle u_3u_1\rangle \;=\; 0\,,$$

i.e.

$$\langle u_i\, u_j\rangle \;=\; u_0^2\,\delta_{ij}\,.$$

Two-point correlations

Two-point velocity correlations retain more information on the structure of turbulence than singlepoint correlations.

In isotropic turbulence, which is also on average invariant under reflection in space, there are only two independent, non-trivial, two-point velocity correlations; the longitudinal covariance, f, defined by,

$$u_0^2\, f(r) \;=\; \Big\langle u(\vec{0},t)\, u(r\vec{e}_x,t)\Big\rangle \;=\; \Big\langle v(\vec{0},t) v(r\vec{e}_y,t)\Big\rangle\;,$$

and the transverse covariance, g, defined by,

$$u_0^2\, g(r) \;=\; \Big\langle v(\vec{0},t)\, v(r\vec{e}_x,t)\Big\rangle \;=\; \Big\langle u(\vec{0},t) u(r\vec{e}_y,t)\Big\rangle\;.$$

The general two-point correlation, $\langle u_i(\vec{x},t)u_j(\vec{x}',t)\rangle$, must be a proper tensor function of u_0^2 and $\vec{r}=\vec{x}'-\vec{x}$. Dependence on any other combination of $\vec{x}$ and $\vec{x}'$ is ruled out by spatial homogeneity. When the turbulence is on average invariant under reflection in space, the two-point correlation can be written in the form,

$$\Big\langle u_i(\vec{x},t)u_j(\vec{x}+\vec{r},t)\Big\rangle \;=\; u_0^2\Big\{g(r)\delta_{ij} + \Big(f(r)-g(r)\Big)\frac{r_ir_j}{r^2}\Big\}\,.$$

The continuity condition;

$$\frac{\partial}{\partial r_j}\Big\langle u_i(\vec{x},t)u_j(\vec{x}{+}\vec{r},t)\Big\rangle = 0 \qquad\Rightarrow\qquad g(r) \;=\; \tfrac{1}{2}r\frac{\mathrm{d}f}{\mathrm{d}r} + f(r)\,.$$

Integral length scales

Using the two-point correlations, we can give the length scale, l_{T}, of the energy-bearing eddies and the turbulence a precise definition. The integral length scales are large length scales, $\Lambda \sim l_{\mathrm{T}}$, defined by,

$$\Lambda_l \;=\; \int_0^\infty f(r)\,\mathrm{d}r \;=\; \frac{1}{\langle u^2\rangle}\int_0^\infty \Big\langle u(\vec{0})u(x\vec{e}_x)\Big\rangle\,\mathrm{d}x,$$

and,

$$\Lambda_t \;=\; \int_0^\infty g(r)\,\mathrm{d}r \;=\; \frac{1}{\langle v^2\rangle}\int_0^\infty \Big\langle v(\vec{0})v(x\vec{e}_x)\Big\rangle\,\mathrm{d}x.$$

In isotropic turbulence the integral scales have the same order of magnitude;

$$g(r) \;=\; \tfrac{1}{2}r\frac{\mathrm{d}f}{\mathrm{d}r} + f(r) \qquad\Rightarrow\qquad \Lambda_t \;=\; \tfrac{1}{2}\Lambda_l\,.$$

Taylor microscales

The Taylor microscales are quite small length scales defined by,

$$\lambda_l \;=\; \left\{\frac{2}{-f''(0)}\right\}^{1/2} \qquad \text{and} \qquad \lambda_t \;=\; \left\{\frac{2}{-g''(0)}\right\}^{1/2}.$$

In isotropic turbulence the Taylor microscales have the same order of magnitude;

$$g(r) \;=\; \tfrac{1}{2} r \frac{df}{dr} + f(r) \qquad \Rightarrow \qquad \lambda_l \;=\; \sqrt{2}\, \lambda_t \,.$$

The Taylor microscales are intermediate length scales, $\quad l_{\mathrm{T}} \;\sim\; \Lambda > \lambda > \eta = l_{\mathrm{K}}$.

The Taylor microscale can be defined more generally, and more loosely *i.e* in an order-of-magnitude fashion, in terms of the correlation of velocity gradients which dominates the dissipation,

$$\left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right\rangle \;\sim\; \frac{u_{\mathrm{T}}^2}{\lambda^2} \qquad \longleftrightarrow \qquad \varepsilon \;\sim\; \frac{\nu u_{\mathrm{T}}^2}{\lambda_t^2}.$$

The expressions for the dissipation which are given in the next subsection show how these relations are satisfied in isotropic turbulence. (A physical understanding of λ in terms of a particular scale of structures in the turbulence is still difficult to achieve since the velocity scale u_{T} in $\varepsilon \sim \nu u_{\mathrm{T}}^2/\lambda_t^2$ is 'wrong' in the sense that it characterizes the large energy-bearing eddies while ν characterize the small dissipative eddies.)

The quasi-equilibrium estimate for ε in high-Re_{T} turbulence yields the scale relations:

$$\varepsilon \;\sim\; \frac{u_{\mathrm{T}}^2}{l_{\mathrm{T}}/u_{\mathrm{T}}} \;=\; \frac{u_{\mathrm{T}}^3}{l_{\mathrm{T}}} \qquad \text{with} \quad \varepsilon \;\sim\; \frac{\nu u_{\mathrm{T}}^2}{\lambda^2} \qquad \longrightarrow \qquad \frac{\lambda}{l_{\mathrm{T}}} \;\sim\; \frac{\lambda}{\Lambda} \;\sim\; \mathrm{Re}_{\mathrm{T}}^{-1/2}.$$

In terms of the Kolomogorov microscales,

$$\varepsilon \;=\; \frac{\nu u_{\mathrm{K}}^2}{l_{\mathrm{K}}^2} \qquad \text{with} \quad \varepsilon \;\sim\; \frac{\nu u_{\mathrm{T}}^2}{\lambda^2} \qquad \longrightarrow \qquad \lambda \;\sim\; \frac{u_{\mathrm{T}}}{u_{\mathrm{K}}}\,\eta \qquad \rightarrow \qquad \frac{\eta}{\lambda} \;\sim\; \mathrm{Re}_{\mathrm{T}}^{-1/4}.$$

Example: The earth's planetary boundary layer ($\mathrm{Re}_{\mathrm{T}} \sim 10^8$);

$$l_{\mathrm{T}} \sim 1\,\mathrm{km}, \qquad \lambda \sim 0{,}1\,\mathrm{m}, \qquad \eta \sim 1\,\mathrm{mm}.$$

Viscous dissipation

In terms of the two-point velocity correlations, the mean rate of viscous dissipation of the kinetic energy of the turbulence is,

$$
\begin{aligned}
\varepsilon &= 2\nu \left\langle s_{ij} \frac{\partial u_i}{\partial x_j} \right\rangle = \nu \left\langle \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} \right\rangle = \nu \left\langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right\rangle \\
&= \nu \left[-\frac{\partial^2}{\partial r_j \partial r_j} \Big\langle u_i(\vec{x}) u_i(\vec{x}+\vec{r}) \Big\rangle \right]_{r=0} \\
&= \nu \left[-3u_0^2 \Big\{ f''(0) + 2g''(0) \Big\} \right] \\
&= 15 \frac{\nu u_0^2}{\lambda_t^2} = 30 \frac{\nu u_0^2}{\lambda_l^2}
\end{aligned}
$$

In the last line the Taylor microscales, λ_l and λ_t, have been introduced.

Taylor's hypothesis

An eddy of lengthscale ℓ will be swept past a probe in a frozen state provided,

$$
t(\ell) \gg \frac{\ell}{U} \quad \longrightarrow \quad u(\ell) \approx \frac{\ell}{t(\ell)} \ll U,
$$

where U is the mean vleocity. For $\mathrm{Re}_\mathrm{T} \gg 1$, and ℓ in the inertial sub-range of scales, this condition implies that,

$$
(\varepsilon \ell)^{1/3} \ll U.
$$

Using the quasi-equilibrium estimate, $\varepsilon \sim u_\mathrm{T}^3 / l_\mathrm{T}$, this condition becomes,

$$
\ell \ll \left(\frac{U}{u_\mathrm{T}} \right)^3 l_\mathrm{T},
$$

which is usually a fairly weak requirement.

It will be possible to construct the two-point covariance $f(r)$ from two-time analysis of the signal from a single probe, and thus measure the longitudinal Taylor microscale λ_l, provided λ_l itself satisfies,

$$
\lambda_\mathrm{l} \ll \left(\frac{U}{u_\mathrm{T}} \right)^3 l_\mathrm{T},
$$

or more generally, including lower Re_T, provided,

$$
u(\lambda_\mathrm{l}) \ll U.
$$

If the eddies of lengthscale λ_l can be assumed to be isotropic the rate of viscous dissipation can now be calculated from $\varepsilon = 30\nu u_0^2/\lambda_l^2$.

Exercise (*inlärningsövning*)

Check that the two-point velocity correlation

$$\Big\langle u_i(\vec{x},t)u_j(\vec{x}+\vec{r},t)\Big\rangle \;=\; u_0^2\left\{g(r)\delta_{ij} + \big(f(r)-g(r)\big)\frac{r_i r_j}{r^2}\right\}$$

reduces to $\big\langle u_i(\vec{x},t)u_j(\vec{x},t)\big\rangle \;=\; u_0^2\,\delta_{ij}$ in the singlepoint limit, $r \to 0$.

Recommended course reading

Sec. 6.3 'Two–point correlation' in "Turbulent flows" by S. B. Pope, C.U.P., 2000.

Alternative recommended reading

"Turbulence", J.O. Hinze, McGraw-Hill (1st edition, 1959; 2nd edition, 1975).

pp 53-56 & 65-66 in Landahl & Mollo-Christensen (1986).

Secs 5.1 & 5.2 in "Introduction to Turbulence" by P.A. Libby (1996).

Sec. 3.1, 'Velocity correlations and spatial scales', in 'An introduction to turbulent flow', J. Mathieu & J. Scott, C.U.P., (2000).

Sections 19.4 and 20.2 in "Physical Fluid Dynamics" by D.J. Tritton, O.U.P., 1988.

B. The wave-number spectrum of isotropic turbulence

Two-point velocity correlations do not readily yield information on length scales which are smaller than the Taylor micro-scales. This information is contained in correlations of two-point velocity differences, $\vec{u}(\vec{x}+\vec{r},t) - \vec{u}(\vec{x},t)$, and in correlations of velocity derivatives. The Fourier transform filters this information out of the two-point velocity correlations. This is one reason for studying the wave-number spectrum.

Another reason is that the dynamical equation governing the two-point correlation is a little bit less difficult to solve in wave-number space, *i.e.* after Fourier-transforming with respect to the separation between the points, $\vec{r}$. This is particularly true of terms which contain the pressure fluctuations, p. (The derivation of the dynamical equation is not included in the examination for this course.)

The Fourier wave-number spectrum of homogeneous turbulence is defined by,

$$\widehat{\Phi}_{ij}(\vec{\kappa},t) \;=\; \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\Big\langle u_i(\vec{x},t)\,u_j(\vec{x}+\vec{r},t)\Big\rangle\,\exp(-\mathrm{i}\vec{\kappa}.\vec{r})\,\mathrm{d}^3\vec{r},$$

so that the inverse formula yields the spectral representation,

$$\Big\langle u_i(\vec{x},t)\,u_j(\vec{x}+\vec{r},t)\Big\rangle \;=\; \int_{-\infty}^{\infty}\widehat{\Phi}_{ij}(\vec{\kappa},t)\,\exp(\mathrm{i}\vec{\kappa}.\vec{r})\,\mathrm{d}^3\vec{\kappa}\,.$$

Remember that small lengthscales correspond to large wave numbers and, *vice versa*, large lengthscales to small wave numbers.

General basic properties

The velocity is real so $[u_i(\vec{x},t)]^* = u_i(\vec{x},t)$ where $[\ldots]^*$ denotes complex conjugation. Now, using the definition of $\widehat{\Phi}_{ij}(\vec{\kappa},t)$ above,

$$\Big[\widehat{\Phi}_{ij}(\vec{\kappa},t)\Big]^* \;=\; \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\Big\langle u_i(\vec{x},t)\,u_j(\vec{x}+\vec{r},t)\Big\rangle^*\,\exp(+\mathrm{i}\vec{\kappa}.\vec{r})\,\mathrm{d}^3\vec{r}$$

$$=\; \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\Big\langle u_i(\vec{x},t)\,u_j(\vec{x}+\vec{r},t)\Big\rangle\,\exp(-\mathrm{i}\,(-\vec{\kappa})\,.\vec{r})\,\mathrm{d}^3\vec{r} \;=\; \widehat{\Phi}_{ij}(-\vec{\kappa},t).$$

The ‘usual’ relation in Fourier analysis.

In statistically homogeneous turbulence the two–point correlation remains unchanged when the position $\vec{x}$ is shifted to another point $\vec{x}'$;

$$\Big\langle u_i(\vec{x},t)\,u_j(\vec{x}+\vec{r},t)\Big\rangle \;=\; \Big\langle u_j(\vec{x}+\vec{r},t)\,u_i(\vec{x},t)\Big\rangle \;=\; \Big\langle u_j(\vec{x}',t)\,u_i(\vec{x}'-\vec{r},t)\Big\rangle,$$

where $\vec{x}' = \vec{x}+\vec{r}$ so that $\vec{x} = \vec{x}'-\vec{r}$. The definition of the Fourier wave-number spectrum implies now that,

$$\begin{aligned}
\widehat{\Phi}_{ij}(\vec{\kappa},t) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\Big\langle u_j(\vec{x}',t)\,u_i(\vec{x}'-\vec{r},t)\Big\rangle\,\exp(-\mathrm{i}\vec{\kappa}.\vec{r})\,\mathrm{d}^3\vec{r} \\
&= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\Big\langle u_j(\vec{x}',t)\,u_i(\vec{x}'+\vec{r}',t)\Big\rangle\,\exp(-\mathrm{i}(-\vec{\kappa}).\vec{r}')\,\mathrm{d}^3\vec{r}' \\
&= \widehat{\Phi}_{ji}(-\vec{\kappa},t),
\end{aligned}$$

using $\vec{\kappa}.\vec{r} = (-\vec{\kappa}).(-\vec{r}) = (-\vec{\kappa}).\vec{r}'$ where $\vec{r}' = -\vec{r}$.

In isotropic turbulence the spectrum is independent of the distinction between $\vec{\kappa}$ and $-\vec{\kappa}$ so,

$$\widehat{\Phi}_{ij}(-\vec{\kappa},t) = \widehat{\Phi}_{ij}(\vec{\kappa},t), \qquad \widehat{\Phi}_{ij}(\vec{\kappa},t) = \widehat{\Phi}_{ji}(\vec{\kappa},t), \quad \text{and} \quad \Big[\widehat{\Phi}_{ij}(\vec{\kappa},t)\Big]^* = \widehat{\Phi}_{ij}(\vec{\kappa},t).$$

The continuity condition

$$\frac{\partial}{\partial r_j}\Big\langle u_i(\vec{x},t)\,u_j(\vec{x}+\vec{r},t)\Big\rangle = 0 \qquad \Rightarrow \qquad \kappa_j\widehat{\Phi}_{ij}(\vec{\kappa},t) = 0.$$

Together with the symmetry property which comes from spatial homogeneity this yields,

$$\kappa_i\widehat{\Phi}_{ij}(\vec{\kappa},t) = 0.$$

Kinetic energy and the singlepoint limit

$$\begin{aligned}
K = \tfrac{1}{2}\Big\langle u_i(\vec{x},t)u_i(\vec{x}+\vec{0},t)\Big\rangle &= \int_{-\infty}^{\infty}\tfrac{1}{2}\widehat{\Phi}_{ii}(\vec{\kappa},t)\,1\,\mathrm{d}^3\vec{\kappa} \\
&= \int_0^{\infty}\oint\tfrac{1}{2}\widehat{\Phi}_{ii}(\vec{\kappa},t)\,\kappa^2\mathrm{d}\Omega_\kappa\,\mathrm{d}\kappa\,,
\end{aligned}$$

when the three-dimensional integral over wave-number space is expressed in spherical coordinates. The mean kinetic energy of the turbulence can be written in the form,

$$K = \int_0^{\infty} E(\kappa,t)\,\mathrm{d}\kappa,$$

where,

$$E(\kappa,t) = \oint\tfrac{1}{2}\widehat{\Phi}_{ii}(\vec{\kappa},t)\,\kappa^2\mathrm{d}\Omega_\kappa\,,$$

is the scalar spectral energy density of the turbulence, given here by its most general definition. For simplicity we will use the scalar spectrum $E(\kappa, t)$ rather than the tensor spectrum $\widehat{\Phi}_{ij}(\vec{\kappa}, t)$ whenever possible.

Isotropic turbulence

The wave-number spectrum, $\widehat{\Phi}_{ij}(\vec{\kappa}, t)$, must be a proper tensor function of $\vec{\kappa}$ that satisfies the continuity condition. In terms of the scalar spectral energy density, $E(\kappa, t)$, it can be written in the form,

$$\widehat{\Phi}_{ij}(\vec{\kappa}, t) \;=\; \frac{E(\kappa, t)}{4\pi\kappa^2}\left\{\delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}\right\},$$

for turbulence that is invariant under spatial reflections. In isotropic turbulence, the scalar spectral energy density is given by,

$$E(\kappa, t) \;=\; 2\pi\kappa^2\,\widehat{\Phi}_{ii}(\vec{\kappa}, t).$$

The Equilibrium range of wave numbers (High $\mathrm{Re_T}$)

When $\mathrm{Re_T} \gg 1$, the small scales are in quasi-equilibrium with the large energy-bearing scales. In terms of the scalar wave-number, κ, the small scales can be defined by,

$$\frac{2\pi}{\kappa} \ll \Lambda \qquad \Rightarrow \qquad \kappa\Lambda \gg 2\pi > 1,$$

i.e. κ is large. This range of wave numbers is characterized by the mean rate of transfer of energy from the large scales to the small scales, *i.e.* by $\varepsilon(t)$. Mathematically, for the scalar spectral energy density, this can be expressed by,

$$E(\kappa, t) \;=\; F_{\mathrm{eq.}}\big(\varepsilon(t), \kappa, \nu\big) \qquad \text{when} \qquad \kappa\Lambda \gg 1.$$

(where 'eq.' stands for 'equilibrium'). (The way in which time, t, appears in this equation gives a concrete example of what is meant by *quasi*-equilibrium.)

The Inertial sub-range of the Equilibrium range (High $\mathrm{Re_T}$)

When $\mathrm{Re_T} \gg 1$, $\Lambda \sim \mathrm{Re_T^{3/4}}\eta \gg \eta$ which means that there will be a substantial range of wave numbers satisfying,

$$\Lambda \gg \frac{2\pi}{\kappa} \gg \eta \qquad \text{or} \qquad \frac{1}{\Lambda} \ll \kappa \ll \frac{1}{\eta}.$$

In this range, $2\pi/\kappa \gg \eta$, *i.e.* $\kappa\eta \ll 1$, implies that viscous processes are negligible (hence the name *inertial*). Mathematically, for the scalar spectral energy density, this means that,

$$E(\kappa, t) \;=\; F_{\mathrm{Ko}}\big(\varepsilon(t), \kappa\big),$$

where 'Ko' stands for the Russian scientist Kolmogorov. Now, dimensional analysis yields,

$$E(\kappa, t) \;=\; \alpha\, \varepsilon^{2/3} \kappa^{-5/3},$$

which is often refered to as the Kolmogorov spectrum.

The Equilibrium range including the Dissipation sub-range (High $\mathrm{Re_T}$)

Using $F_{\mathrm{Ko}} = \alpha\, \varepsilon^{2/3}\kappa^{-5/3}$ to express $F_{\mathrm{eq.}}$,

$$E(\kappa, t) \;=\; \alpha\, \varepsilon^{2/3} \kappa^{-5/3} f_{\mathrm{eq.}}(\kappa\eta),$$

in the whole equilibrium range, $\kappa\Lambda \gg 1$, which can be divided into sub-ranges according to;.

the inertial sub-range	$\kappa \ll \kappa_{\mathrm{d}}$	$\kappa\eta \simeq 0$ and $f_{\mathrm{eq.}}(\kappa\eta) \simeq 1$
the dissipation sub-range	$\kappa \simeq \kappa_{\mathrm{d}}$	$2\nu\kappa^2 E$ has a maximum at $\kappa\eta < 1$
the far dissipation sub-range	$\kappa \gg \kappa_{\mathrm{d}}$	$f_{\mathrm{eq.}} \to 0$ exponentially when $\kappa\eta \to \infty$

where $\kappa_{\mathrm{d}} = 2\pi/\eta = \kappa_{\mathrm{Ko}}$.

Comparisons with measurements

The measurements required for the evaluation of three-dimensional spectra, such as $E(\kappa)$, are much more demanding than those required for one-dimensional spectra. Consequently comparisons with measurements are usually based on the one-dimensional spectrum,

$$u_0^2\, \hat{f}(\kappa_x) \;=\; \frac{1}{\pi} \int_0^\infty u_0^2\, f(x)\, \cos(\kappa_x x)\, \mathrm{d}x\,,$$

where the longitudinal covariance, $f(x)$, was defined on page 2. This two-point correlation can be constructed from two-time analysis of the signal from a single probe using Taylor's hypothesis.

In the inertial sub-range of the equilibrium range of lengthscales or wavenumbers,

$$u_0^2\, \hat{f}(\kappa_x) \;=\; \alpha'\, \varepsilon^{2/3} \kappa_x^{-5/3},$$

and in isotropic turbulence,

$$\alpha' \;=\; \frac{18}{55}\,\alpha \;=\; 0.33\,\alpha.$$

Measurements indicate that $\alpha = 1.5$.

Dissipation rate

In terms of the scalar spectral energy density, the viscous dissipation of kinetic energy of homogeneous turbulence is given by,

$$\begin{aligned}\varepsilon \;&=\; \nu\left[-\frac{\partial^2}{\partial r_j \partial r_j}\,\langle u_i(\vec{x},t)u_i(\vec{x}+\vec{r},t)\rangle\right]_{r=0} \\ &=\; \int_0^\infty 2\nu\kappa^2\,E(\kappa,t)\,\mathrm{d}\kappa\end{aligned}$$

Recommended course reading

Sec. 13.9, 'Spectrum of turbulence in inertial subrange', in "Fluid Mechanics", by P.K. Kundu & I.M. Cohen.

Sec. 6.5 'Velocity spectra' in "Turbulent flows" by S. B. Pope, C.U.P., 2000.

Secs 8.1-8.4 in Tennekes & Lumley (1972).

Alternative recommended reading

"A Model of Turbulence", Leo P. Kadanoff, Physics Today, September 1995

pp 59-64 in Landahl & Mollo-Christensen (1986).

Sec. 5.2 in "Introduction to Turbulence" by P.A. Libby (1996).

Ch. 6, 'Spectral analysis of homogeneous turbulence', pp 239–245, in 'An introduction to turbulent flow', J. Mathieu & J. Scott, C.U.P., (2000).

Sec. 6.4, 'Consequences of isotropy', in 'An introduction to turbulent flow', J. Mathieu & J. Scott, C.U.P., (2000).

Sections 19.5 and 20.3 in "Physical Fluid Dynamics" by D.J. Tritton, O.U.P., 1988.

C. The Dynamical Equation for the Energy Spectrum

The dynamical equation for $E(\kappa, t)$ in isotropic turbulence can be derived in a straightforward way from the dynamical equation for the two-point velocity correlation *via* the dynamical equation for $\widehat{\Phi}_{ij}(\vec{\kappa}, t)$. See the appendix. The resulting equation is written in the form,

$$\frac{\partial E}{\partial t} \;=\; T - 2\nu\kappa^2 E.$$

If instead of isotropic turbulence we consider homogeneous turbulence under the influence of constant mean shear the equation becomes,

$$\frac{\partial E}{\partial t} \;=\; \mathcal{P} + T - 2\nu\kappa^2 E,$$

where $\mathcal{P}(\kappa)$ is the production at scalar wavenumber κ. In D.N.S., $\mathcal{P}(\kappa)$ is given by the forcing.

$2\nu\kappa^2 E(\kappa, t)$ is the rate of viscous dissipation of energy at scalar wave number κ — see the end of sec. B.

$T(\kappa, t)$ is the Fourier transform of the terms containing third-order correlations in the two-point equation. The detailed expression for $T(\kappa, t)$ is a bit complicated and contains the third-order spectrum which has not been defined in these notes. Since the terms containing third-order correlations in the two-point equation vanish in the singlepoint limit, *i.e.* when $r \to 0$, $T(\kappa, t)$ satisfies,

$$\int_0^\infty T(\kappa, t)\,\mathrm{d}\kappa \;=\; 0.$$

$T(\kappa, t)$ is the net effect on $E(\kappa, t)$ of the non-linear inertial spectral transfer of the mean kinetic energy of the turbulence from large scales (small κ) to small scales (large κ). Consequently, $T(\kappa, t)$ is negative for small κ and positive for large κ. In terms of the spectral transfer of energy, ε_f, the transfer out of the energy-bearing eddies (small κ) is,

$$\int_0^{\kappa_{\text{inert}}} T(\kappa, t)\,\mathrm{d}\kappa \;\simeq\; -\varepsilon_f\,.$$

where κ_{inert} is a wave-number, any wave-number, in the inertial sub-range of the equilibrium range. Well into the quasi-equilibrium range, $\kappa \geq \kappa_{\text{inert}}$, we expect,

$$2\nu\kappa^2 E(\kappa, t) \;\simeq\; T(\kappa, t)\;,$$

so that,

$$\int_{\kappa_{\text{inert}}}^\infty T(\kappa, t)\,\mathrm{d}\kappa \;\simeq\; \varepsilon,$$

is the transfer into the dissipating eddies (large κ).

The equation for $E(\kappa, t)$ is not closed. $T(\kappa, t)$ has to be modelled in terms of $E(\kappa, t)$ before $E(\kappa, t)$ can be calculated.

The singlepoint limit

The singlepoint limit, $r \to 0$, is achieved by integrating over all wave numbers;

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}K \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty E\,\mathrm{d}\kappa \;&=\; \int_0^\infty \frac{\partial E}{\partial t}\,\mathrm{d}\kappa \\
&=\; \int_0^\infty T\,\mathrm{d}\kappa \;-\; \int_0^\infty 2\nu\kappa^2 E\,\mathrm{d}\kappa \\
&=\; 0 \;-\; \varepsilon\,.
\end{aligned}
$$

This is just the K-equation that was obtained for homogeneous turbulence in the absence of mean shear in Chapter 5 of these notes.

Recommended course reading

Sec. 6.6 'The spectral view of the energy cascade' in "Turbulent flows" by S. B. Pope, C.U.P., 2000.

Alternative recommended reading

Sec. 6.3, 'Spectral equations via correlations in physical space', in 'An introduction to turbulent flow', J. Mathieu & J. Scott, C.U.P., (2000).

Sections 20.2 and 20.3 in "Physical Fluid Dynamics" by D.J. Tritton, O.U.P., 1988.

Appendix. The Derivation of the Dynamical Equations

The dynamical equation for $E(\kappa, t)$ in isotropic turbulence can be derived in a straightforward way from the dynamical equation for the two-point velocity correlation *via* the dynamical equation for $\widehat{\Phi}_{ij}(\vec{\kappa}, t)$. The *derivation* of the dynamical equation is not included in the examination for this course.

The two-point dynamical equation

In the absence of a mean flow, the fluctuating instantaneous velocity satisfies,

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_l} u_i u_l + \frac{\partial}{\partial x_l} \langle u_i u_l \rangle - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_l \partial x_l}.$$

Together with,

$$\frac{\partial}{\partial t} \Big\langle u_i(\vec{x}_1, t)\, u_j(\vec{x}_2, t) \Big\rangle = \Big\langle \frac{\partial u_i}{\partial t}(\vec{x}_1, t)\, u_j(\vec{x}_2, t) \Big\rangle + \Big\langle u_i(\vec{x}_1, t)\, \frac{\partial u_j}{\partial t}(\vec{x}_2, t) \Big\rangle$$

this leads straightforwardly to,

$$\begin{aligned} \frac{\partial}{\partial t} \Big\langle u_i(\vec{x}_1, t)\, u_j(\vec{x}_2, t) \Big\rangle &= \\ &- \frac{\partial}{\partial x_{1l}} \Big\langle u_i(\vec{x}_1, t)\, u_j(\vec{x}_2, t)\, u_l(\vec{x}_1, t) \Big\rangle - \frac{\partial}{\partial x_{2l}} \Big\langle u_i(\vec{x}_1, t)\, u_j(\vec{x}_2, t)\, u_l(\vec{x}_2, t) \Big\rangle \\ &- \frac{\partial}{\partial x_{1i}} \Big\langle \frac{1}{\rho} p(\vec{x}_1, t)\, u_j(\vec{x}_2, t) \Big\rangle - \frac{\partial}{\partial x_{2j}} \Big\langle \frac{1}{\rho} p(\vec{x}_2, t)\, u_i(\vec{x}_1, t) \Big\rangle \\ &+ \nu \left(\frac{\partial^2}{\partial x_{1l} \partial x_{1l}} + \frac{\partial^2}{\partial x_{2l} \partial x_{2l}} \right) \Big\langle u_i(\vec{x}_1, t)\, u_j(\vec{x}_2, t) \Big\rangle. \end{aligned}$$

This equation is subjected to a coordinate transformation, $\{\vec{x}_1, \vec{x}_2\} \to \{\vec{r}, \vec{x}\}$, where,

$$\left.\begin{aligned} \vec{x}_1 &= \vec{x} \\ \vec{x}_2 &= \vec{x} + \vec{r} \end{aligned}\right\} \iff \left\{\begin{aligned} \vec{r} &= \vec{x}_2 - \vec{x}_1 \\ \vec{x} &= \vec{x}_1 \end{aligned}\right.$$

so that $\vec{x}$ is a 'position' and $\vec{r}$ is the separation between the two original points. The spatial derivatives are given by

$$\frac{\partial}{\partial x_{1l}} = -\frac{\partial}{\partial r_l} + \frac{\partial}{\partial x_l} \qquad \text{and} \qquad \frac{\partial}{\partial x_{2l}} = \frac{\partial}{\partial r_l}$$

In spatially homogeneous turbulence, averaged quantities depend on $\vec{r}$ but not on $\vec{x}$ ('$\frac{\partial}{\partial x_l} = 0$') so that the two-point dynamical equation becomes,

$$\frac{\partial}{\partial t}\left\langle u_i(\vec{x},t)\, u_j(\vec{x}+\vec{r},t)\right\rangle =$$

$$\frac{\partial}{\partial r_l}\left\langle u_i(\vec{x},t)\, u_j(\vec{x}+\vec{r},t)\, u_l(\vec{x},t)\right\rangle - \frac{\partial}{\partial r_l}\left\langle u_i(\vec{x},t)\, u_j(\vec{x}+\vec{r},t)\, u_l(\vec{x}+\vec{r},t)\right\rangle$$

$$+ \frac{\partial}{\partial r_i}\left\langle \frac{1}{\rho} p(\vec{x},t)\, u_j(\vec{x}+\vec{r},t)\right\rangle - \frac{\partial}{\partial r_j}\left\langle \frac{1}{\rho} p(\vec{x}+\vec{r},t)\, u_i(\vec{x},t)\right\rangle$$

$$+ 2\nu \frac{\partial^2}{\partial r_l \partial r_l}\left\langle u_i(\vec{x},t)\, u_j(\vec{x}+\vec{r},t)\right\rangle$$

The presence of third-order velocity correlations in the dynamical equation for the second-order correlation is an example of the general closure problem.

In the singlepoint limit, *i.e.* when $\quad r \to 0$,

$$\frac{\partial}{\partial r_l}\left\langle u_i(\vec{x},t)\, u_j(\vec{x}+\vec{r},t)\, u_l(\vec{x},t)\right\rangle - \frac{\partial}{\partial r_l}\left\langle u_i(\vec{x},t)\, u_j(\vec{x}+\vec{r},t)\, u_l(\vec{x}+\vec{r},t)\right\rangle \quad \to \quad 0.$$

(The Fourier transform of this expression is the rate of transfer of energy through spectral space, $T(\kappa,t)$, and the fact that it vanishes in the singlepoint limit leads to $\int_0^\infty T\, \mathrm{d}\kappa = 0$.)

Exercises (*inlärningsövningar*)

Show that $\quad \left\langle \rho^{-1} p(\vec{x},t)\, u_j(\vec{x}+\vec{r},t)\right\rangle = 0 \quad$ in isotropic turbulence.

Show that, in homogeneous turbulence, the contracted forms of the pressure terms vanish;

$$\frac{\partial}{\partial r_i}\left\langle \frac{1}{\rho} p(\vec{x},t)\, u_i(\vec{x}+\vec{r},t)\right\rangle = 0,$$

and,

$$\frac{\partial}{\partial r_i}\left\langle \frac{1}{\rho} p(\vec{x}+\vec{r},t)\, u_i(\vec{x},t)\right\rangle = 0.$$

The dynamical equation for the energy spectrum

In isotropic turbulence, the energy spectrum, E, can be written in terms of the two-point velocity correlation in the form,

$$E(\kappa, t) \;=\; 2\pi\kappa^2 \widehat{\Phi}_{ii}(\vec{\kappa}, t) \;=\; \frac{2\pi\kappa^2}{(2\pi)^3} \int_{-\infty}^{\infty} \langle u_i(\vec{x}, t)\, u_i(\vec{x}+\vec{r}, t)\rangle \;\; \exp(-\mathrm{i}\vec{\kappa}.\vec{r})\, \mathrm{d}^3\vec{\kappa}\,,$$

so the dynamical equation governing E can be obtained by first contracting ($i = j \Rightarrow$ sum) and then Fourier transforming the equation for the two-point velocity correlation,

$$\frac{\partial}{\partial t} E \;=\; \frac{2\pi\kappa^2}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \langle u_i(\vec{x}, t)\, u_i(\vec{x}+\vec{r}, t)\rangle \;\; \exp(-\mathrm{i}\vec{\kappa}.\vec{r})\, \mathrm{d}^3\vec{\kappa}\,.$$

The resulting equation is written in the form,

$$\frac{\partial E}{\partial t} \;=\; T - 2\nu\kappa^2 E.$$

The detailed expression for $T(\kappa, t)$ is a bit complicated and contains the third-order spectrum which has not been defined in these notes.